Fiber bundles and topology for condensed matter systems

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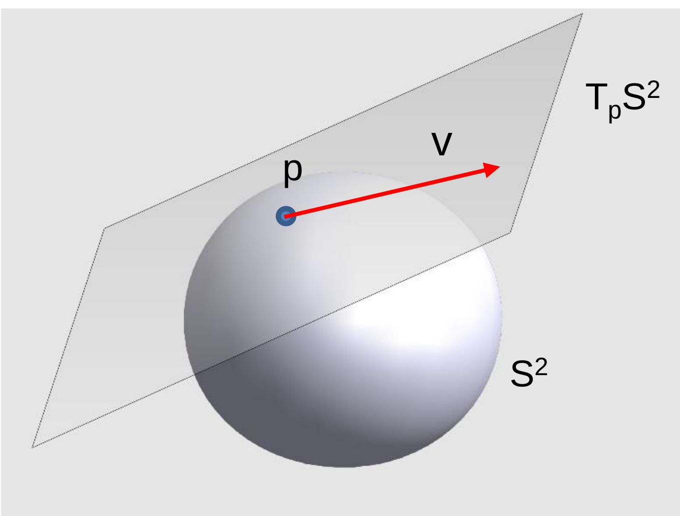
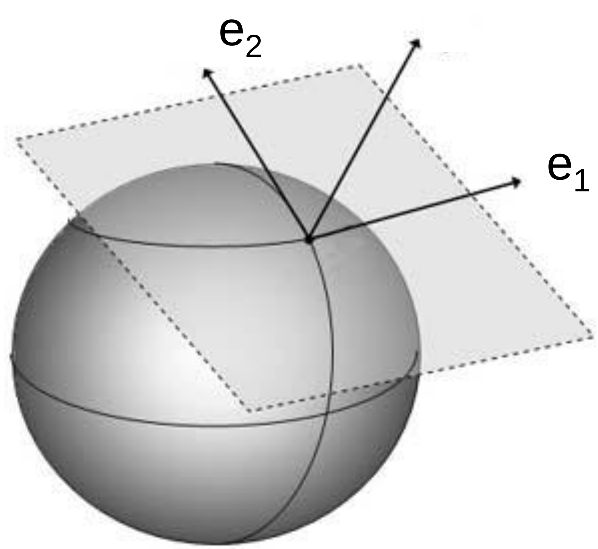
1. Tangent bundles and curvature

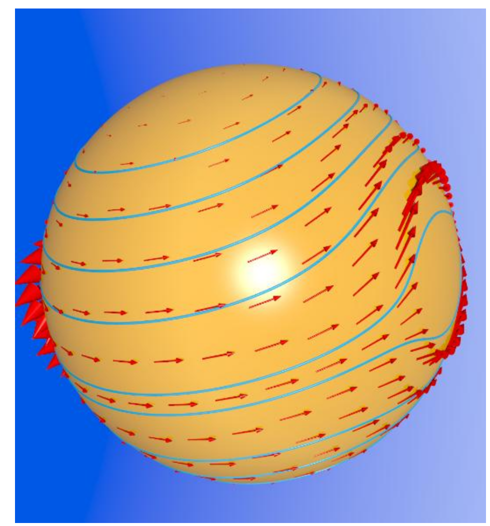
The tangent bundle of the sphere

Sphere S^2: prominent example for a differential manifold

- Sphere plus tangent planes: tangent bundle TS^2
- TS^2 is example for a fiber bundle:
  Basis manifold M is S^2, fibers are the tangent planes T_pS^2
- Point in TS^2: ($p \in S^2$, $v \in T_pS^2$, $v = v^1 e_1 + v^2 e_2$)
- Vector field: "section" of a fiber bundle

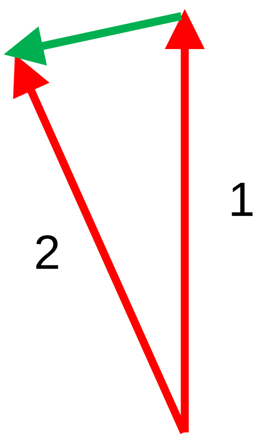


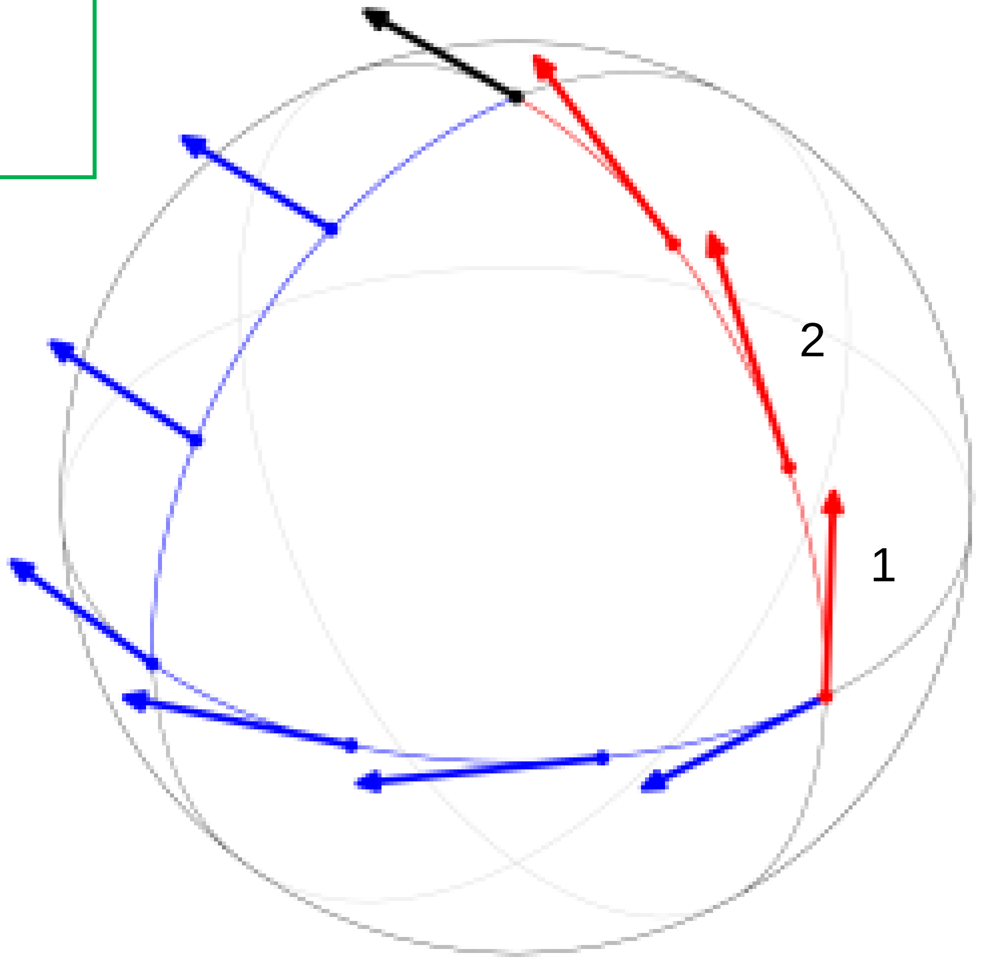




Parallel transport of vectors on S^2

- Comparison of different tangent spaces by parallel transport along a path
- Levi Civita connection





Covariant derivative

- Covariant derivative:

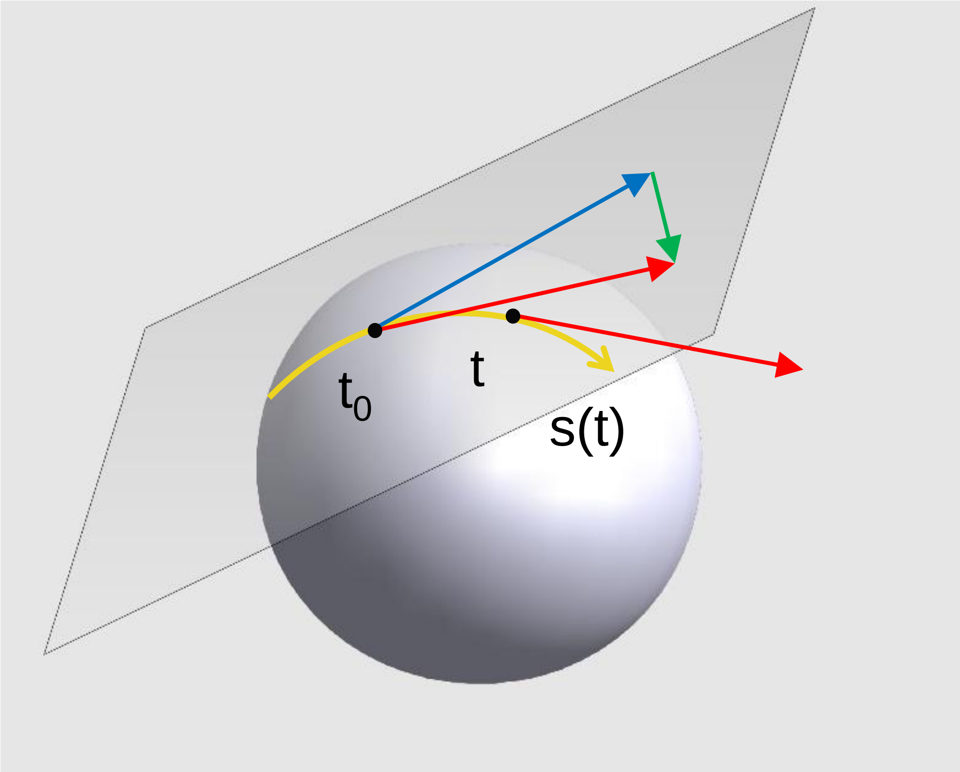
$$u^{\lambda} D_{\lambda} w = \lim_{t \to t_0} \frac{1}{t - t_0} \left[P(t \to t_0) w(t) - w(t_0) \right]$$

$$u = \frac{ds}{dt}(t_0)$$

$$D_{\lambda} w := e_{\mu} \left(\partial_{\lambda} w^{\mu} + \Gamma^{\mu}_{\nu\lambda} w^{\nu} \right)$$

$\Gamma^{\mu}_{\nu\lambda}$: Connection coefficien ts

Parallel transport : $D_{\lambda} w = 0$



Curvature of S^2

- Curvature: transport along closed path rotates vector (holonomy)
- Rotation angle:

$$\Delta\omega = \int_{\text{encircled area}} d\Omega \, \kappa(\theta,\varphi)$$

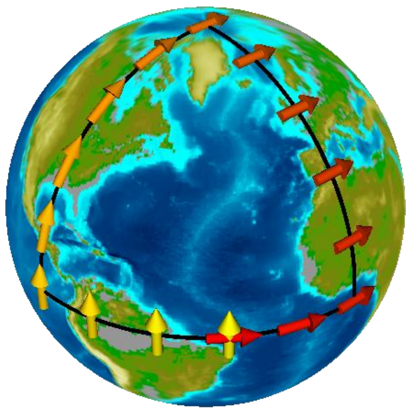
$\kappa =$ Gaussian curvature

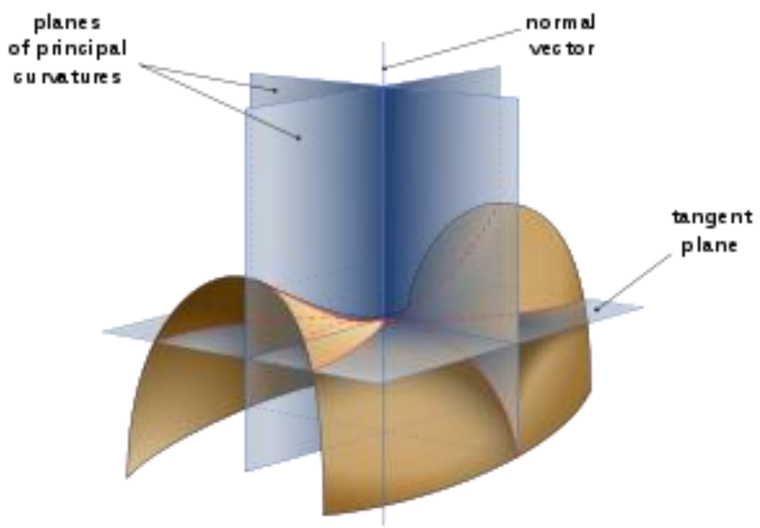
$$\kappa = K_1 K_2 = \frac{1}{R_1}\frac{1}{R_2}$$

- Curvature tensor two-dimensional manifold:

$$R^{\rho}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\lambda\nu} - \partial_{\nu}\Gamma^{\rho}_{\lambda\mu}$$

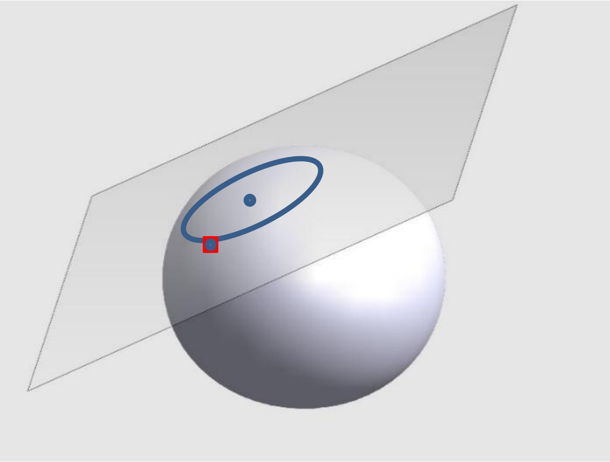
$$R^{1}{}_{212} = -R^{1}{}_{221} = -R^{2}{}_{112} = R^{2}{}_{121} = \kappa$$





Principal bundle

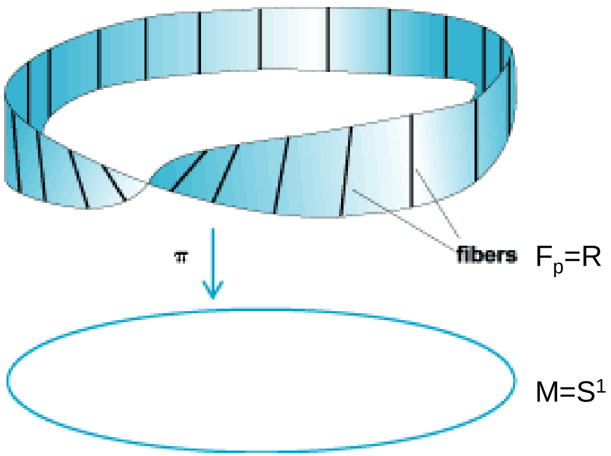
- For description of parallel transport and curvature: attach rotation group SO(2) at each point, yields principal bundle $(S^2, SO(2)) = (S^2, S^1)$
- Change of origin (unit operation): gauge transformation



2. Fiber bundles

Fiber bundles

- Consist of a basic differential manifold M
- At each point attached: fiber F_p which is either copy of a vector space ("vector bundle") or of a (gauge) group ("principle bundle")
- Prescriptions for glueing the fibers together e.g. Moebius strip (S^1,R)



Fiber bundles

- Prescription for parallel transport : covariant derivative and curvature

$$D_\lambda w := e_j \left(\partial_\lambda w^j + \Gamma^j_{i\lambda} w^i \right)$$

$$\underline{\underline{D}}_\lambda \underline{w} = \left(\partial_\lambda + \underline{\underline{\Gamma}}_\lambda \right) \underline{w}, \quad \underline{w} = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$$

$\underline{\underline{\Gamma}}_\lambda$: Connection coefficient matrix

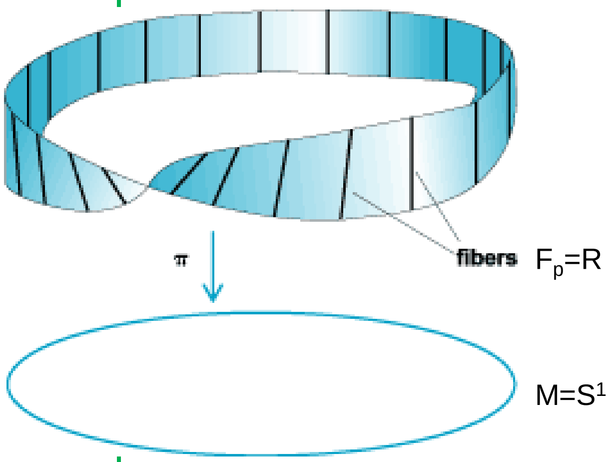
Curvature tensor :

$$R^i_{j\mu\nu} = \partial_\mu \Gamma^i_{j\nu} - \partial_\nu \Gamma^i_{j\mu} + \Gamma^i_{l\mu}\Gamma^l_{j\nu} - \Gamma^i_{l\nu}\Gamma^l_{j\mu}$$

$$\underline{\underline{R}}_{\mu\nu} = \partial_\mu \underline{\underline{\Gamma}}_\nu - \partial_\nu \underline{\underline{\Gamma}}_\mu + \left[\underline{\underline{\Gamma}}_\mu, \underline{\underline{\Gamma}}_\nu \right]$$

- Topological quantum numbers, e.g. for 2d tangent bundles: Euler characteristic:

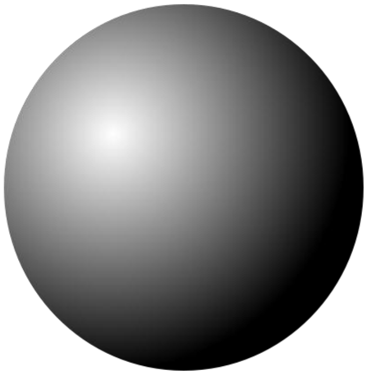
$$\chi = \frac{1}{2\pi} \int_{M^2} d \quad \kappa(\theta,\varphi) = 2 - 2g$$



3. Topological quantum numbers

Euler characteristic

$\chi_E=2$



$\chi_E=0$

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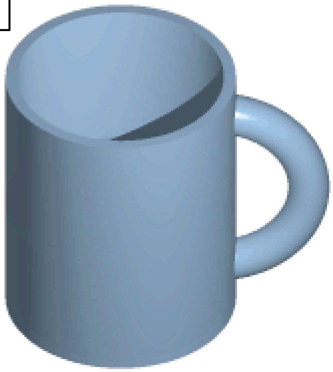
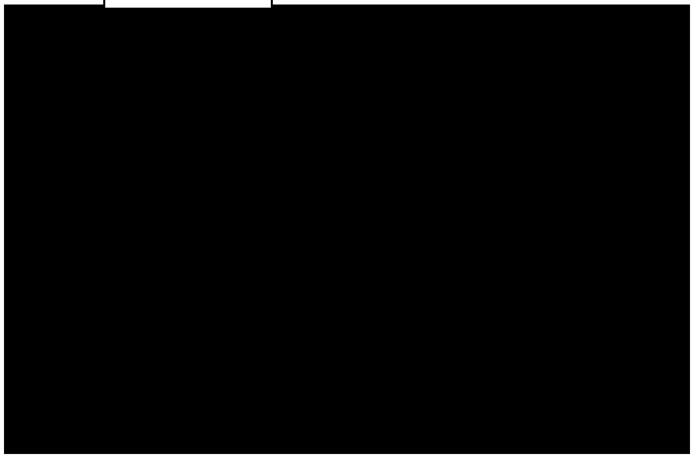


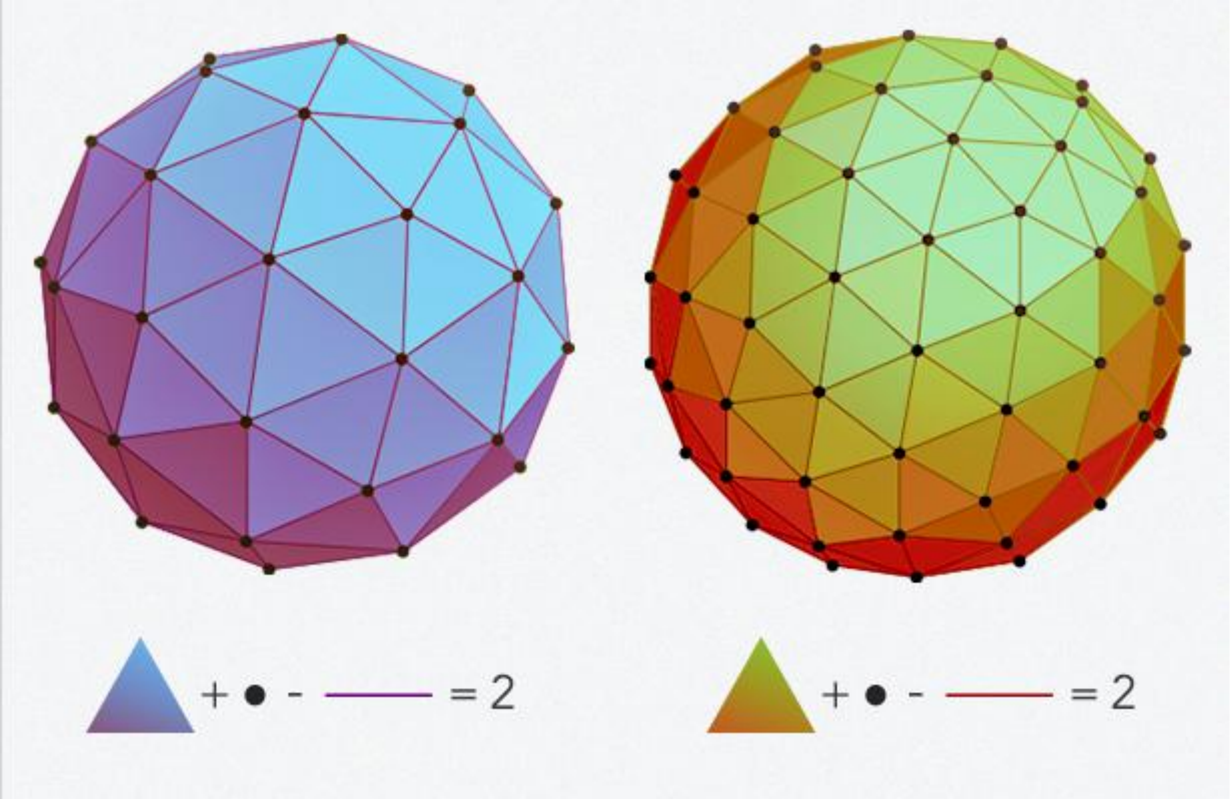
$\chi_E=-2$

$\chi_E=-4$

$\chi_E=-4$



Euler characteristic



$$\chi_E = \#\,\text{faces} + \#\,\text{vertices} - \#\,\text{edges}$$

4. Applications of fiber bundles

Fiber bundle in cosmology

TM^4 or $(M^4, SO(1,3))$, M^4 spacetime

Fiber bundle in electrodynamics

Classical relativistic physics:

- Four momentum : $p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix}$, four velocity : $v^\mu = p^\mu / m$
- In electromagnetic field : $v^\mu = \frac{1}{m}\left(p^\mu - qA^\mu\right), A^\mu = \begin{bmatrix} \Phi \\ \vec{A} \end{bmatrix}$

Quantum mechanics:

- $\psi \in C$, i.e. section through a $(R \otimes R^3, C)$-bundle
- Parallelism?

$$p_\mu \to \frac{\hbar}{i}\partial_\mu, \quad v_\mu = \frac{\hbar}{im}\left(\partial_\mu - i\frac{q}{\hbar}A_\mu\right) \equiv \frac{\hbar}{im}D_\mu$$

$$D_\mu \psi = 0 \Rightarrow \psi = \psi_0 \exp\underbrace{\left\{i\frac{q}{\hbar}\int dx^\nu A_\nu\right\}}_{\text{phase change} \in U(\,)}$$

- $(R \otimes R^3, U(\,))$ principle bundle

Topology:

- Curvature equals the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{cp.} \quad \underline{\underline{R}}_{\mu\nu} = \partial_\mu \underline{\underline{\Gamma}}_\nu - \partial_\nu \underline{\underline{\Gamma}}_\mu + \left[\underline{\underline{\Gamma}}_\mu, \underline{\underline{\Gamma}}_\nu\right]$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

- Topological quantum number : First Chern number

$$\mathrm{ch}(\,) = \frac{1}{2\pi}\underbrace{\int dx^1 dx^2 F_{12}}_{\text{2d closed submanifold}}$$

Magnetic monopole of strength γ

- Vector potential and electromagnetic field tensor (acting on C)

  $-iA_\theta = 0, \quad -iA_\varphi = \gamma i \cos\theta, \quad iF_{\theta\varphi} = \gamma i \sin\theta$

- Cp. Levi-Civita connection on the sphere in orthonormal basis (acting on R^2)

  $$\hat{\underline{\underline{\Gamma}}}_\theta = 0, \quad \hat{\underline{\underline{\Gamma}}}_\varphi = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i} \cos\theta, \quad \hat{\underline{\underline{R}}}_{\theta\varphi} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i} \sin\theta$$

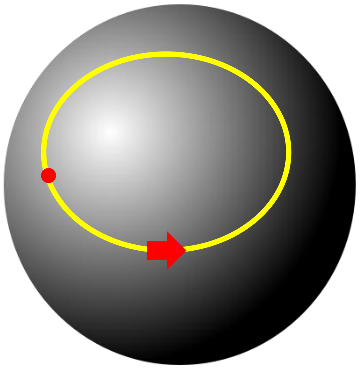
- First chern number : $\mathrm{ch}() = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, F_{\theta\varphi} = 2\gamma$

- $(R \otimes R^3, U(1))$ Electromagnetism
- $(R \otimes R^3, U(1) \otimes SU(2))$ Electroweak interaction
- $(R \otimes R^3, SU(3))$ Strong interaction

5. Topological quantum states

The Berry phase

- Quantum mechanical ground states dependent on parameter $\xi : |m(\xi)\rangle$
- $\xi \in M$ parameter manifold
- Example : Spin - 1/2 – particle in magnetic field of fixed strength, but arbitrary orientation : $M = S^2$, $\xi = (\theta, \varphi)$
- $|m(\xi)\rangle$ is determined up to a phase factor $e^{i\alpha}$, system is principle bundle $(M, U(1))$



The Berry phase

- Adiabatic motion and energy gap:
  particles remain in instantaneous state $\left|m(\lambda)\right\rangle$

- Circular motion: phase change, holonomy

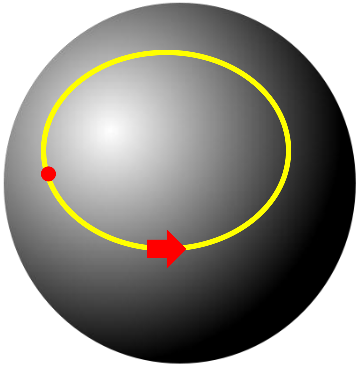
- Covariant derivative $D_\mu = \partial_\mu - iA_\mu$ with
  Berry connection $A_\mu = i\left\langle m \middle| \partial_\mu m \right\rangle$
  Berry curvature $F_{\mu\nu} = i\left(\left\langle \partial_\mu m \middle| \partial_\nu m \right\rangle - \left\langle \partial_\nu m \middle| \partial_\mu m \right\rangle\right)$

- First chern number for 2d parameter space

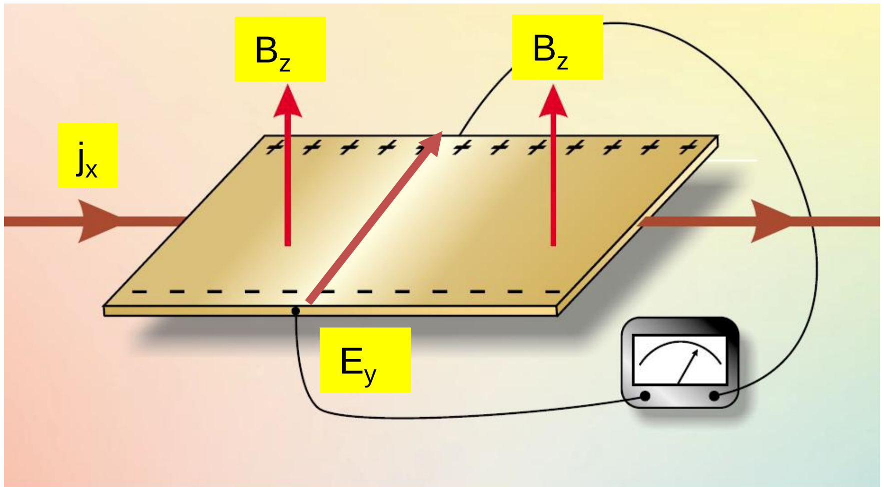
$$ch(M) = \frac{1}{2\pi}\int_M dx^1 dx^2 F_{12}$$

- Example spin-1/2-particle:

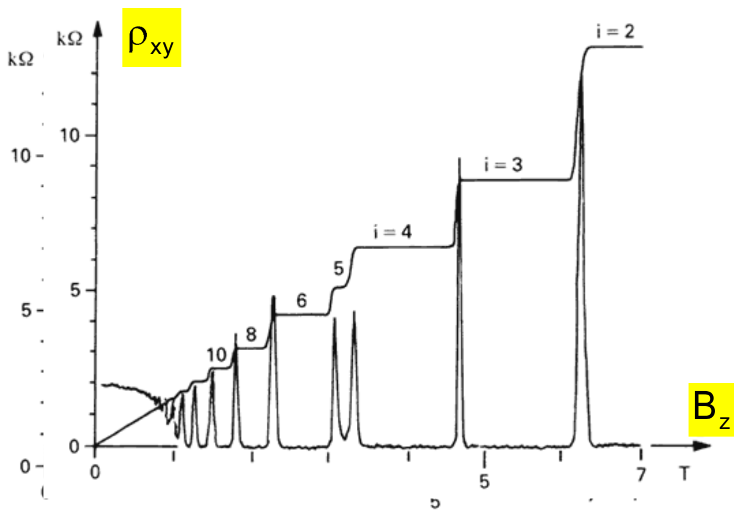
$$-iA_\theta = 0,\ -iA_\varphi = i\frac{1}{2}\cos\theta,\ iF_{\theta\varphi} = i\frac{1}{2}\sin\theta,\ ch() = 1$$

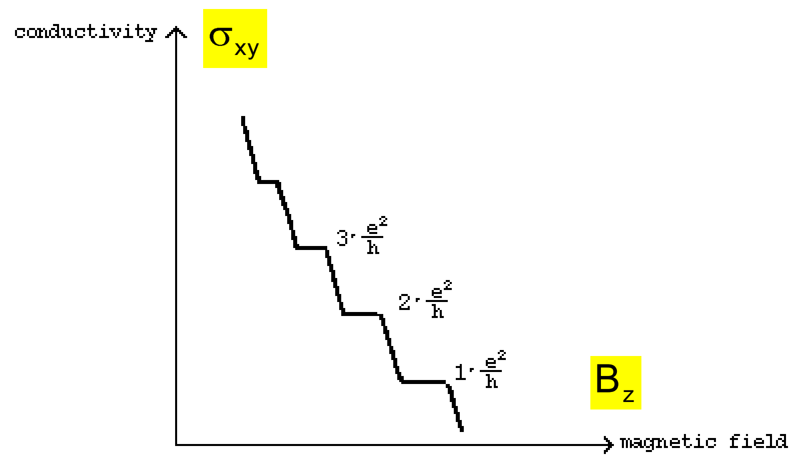


The Quantum Hall Effect (von Klitzing 1980)



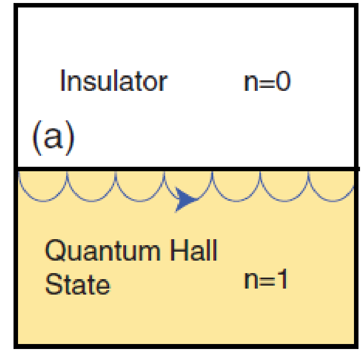
- 2d crystal in magnetic field
- $E_y = \rho_{yx} j_x \qquad j_x = \sigma_{xy} E_y$
- $\rho_{yx} = \frac{1}{n}\frac{h}{e^2} \qquad \sigma_{xy} = \frac{e^2}{h} n$





The Quantum Hall Effect

- Wave functions labelled by wavevector $\vec{k} = (k_1, k_2)$ $|u(\vec{k})\rangle$
- Wavevector space is also periodic : $\vec{k} \in R^2 \bmod (\vec{K}_1, \vec{K}_2) = T^2$
- Principle bundle $(T^2, U(1))$



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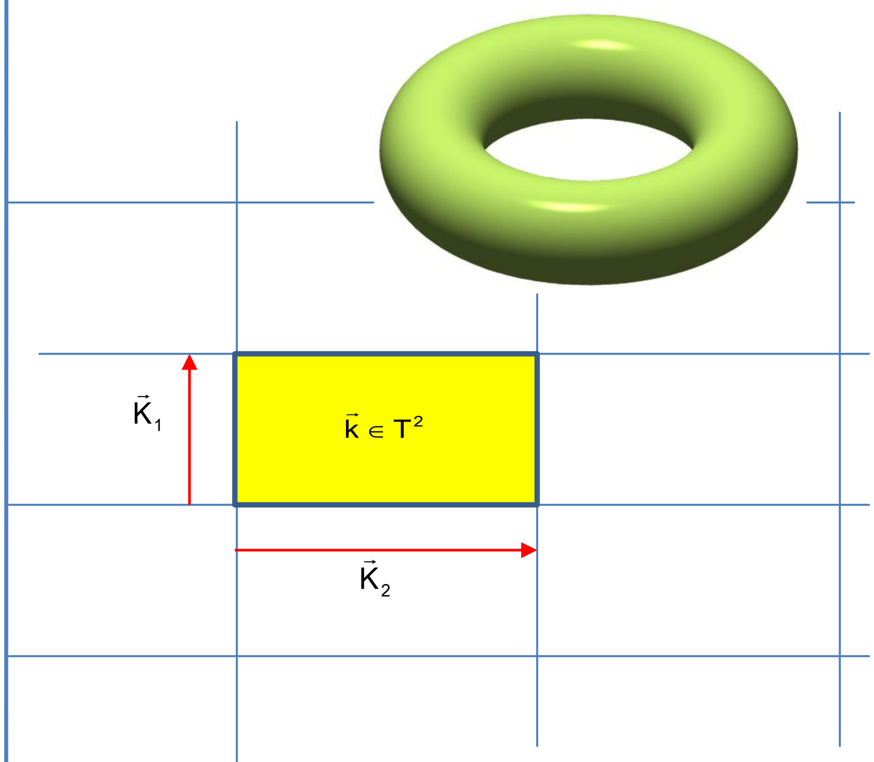


- Kubo transport formula for Hall conductivity :

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\text{occupied bands}} \frac{1}{2\pi} \int_{\text{Torus}} d^2k \; \underbrace{i\left(\langle \partial_1 u | \partial_2 u \rangle - \langle \partial_2 u | \partial_1 u \rangle\right)}_{\text{Berry curvature } F_{12}}$$

$$\sigma_{xy} = \frac{e^2}{h} \underbrace{\sum_{\text{occupied bands}} ch_1(T^2, U(1))}_{n}$$

6. Basics for general topological classification

Search for topological insulators

- Insulator : described by a mapping

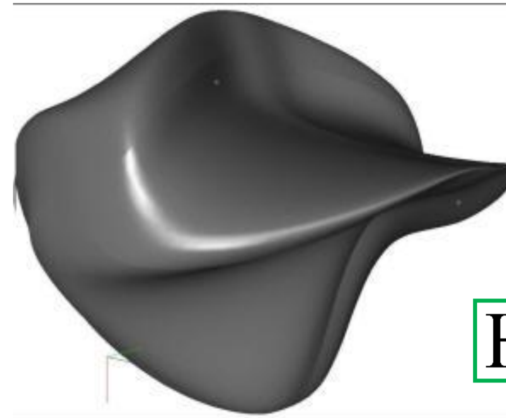
$$T^d \to \mathrm{H}$$
$$\vec{k} \mapsto H(\vec{k})$$

- H? Set of Bloch - Hamiltonians with gap.

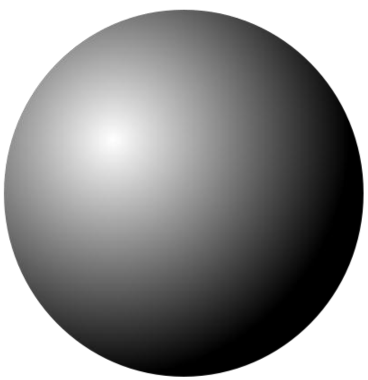
- Set $\pi(T^d, \mathrm{H})$ of topologically different mappings?

- Procedure : Replace H by an equivalent mathematically known standard space K such that

$$\pi(T^d, \mathrm{H}) = \pi(T^d, \mathrm{K})$$



H



K

Topological classification of insulators 1

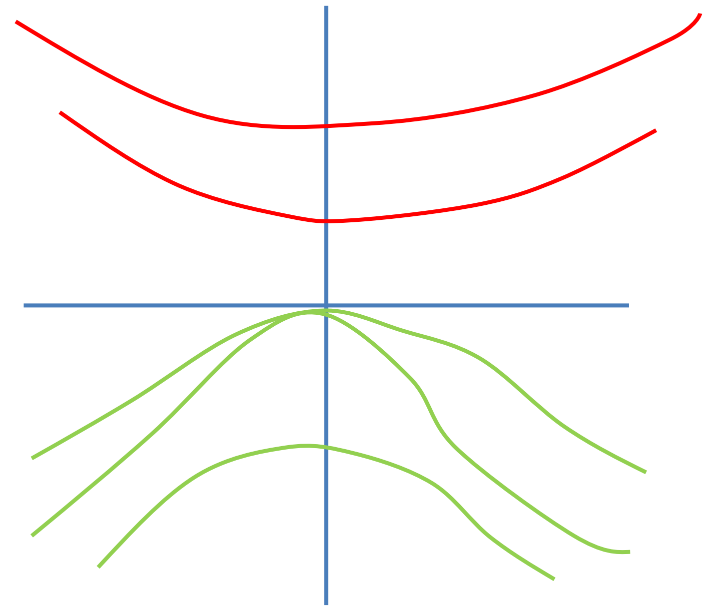
- First step : diagonalize Bloch - Hamilton matrices :

$$\underline{\underline{H}}(\ ) = \langle r | H(\ ) | s \rangle \rightarrow \quad |i\rangle, |j\rangle \text{ eigenstates of } H(\ )$$

$$\underline{\underline{\Delta}}(\ ) = \langle i | H(\ ) | j \rangle = \langle i | r \rangle \langle r | H(\ ) | s \rangle \langle s | j \rangle = \underline{\underline{U}}^{+} \underline{\underline{H}}(\ ) \underline{\underline{U}} =$$

$$= \begin{bmatrix} \varepsilon_m^{+}(\ ) & & & & & \\ & \ddots & & & & \\ & & \varepsilon_1^{+}(\ ) & & & \\ & & & \varepsilon_1^{-}(\ ) & & \\ & & & & \ddots & \\ & & & & & \varepsilon_n^{-}(\ ) \end{bmatrix}$$

- $\underline{\underline{U}} = \underline{\underline{U}}(\ ) \in U(n+m)$
- $\underline{\underline{H}}(\ ) = \underline{\underline{U}}\,\underline{\underline{\Delta}}(\ )\,\underline{\underline{U}}^{+}$



Topological classification of insulators 2

- Second step : deform band structure without closing gap

$$\varepsilon_j^{+}(k) \to +1, \quad \varepsilon_i^{-}(k) \to -1$$

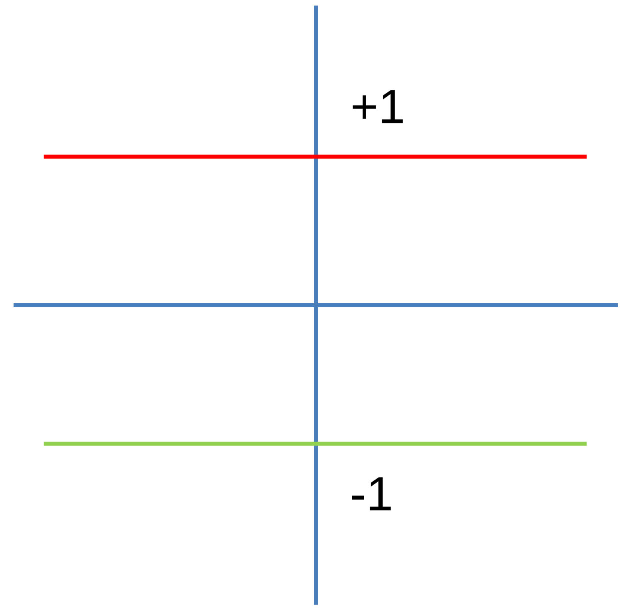
$$\underline{\underline{K}}(k) = \underline{\underline{U}} \begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix} \underline{\underline{U}}^{+} \equiv \underline{\underline{U}}\, \underline{\underline{D}}_n^m\, \underline{\underline{U}}^{+} \in \mathrm{K}$$

- K is "Orbit" of $\underline{\underline{D}}_n^m$ under the action of the group $U(m+n)$
- Fixpoint group of $\underline{\underline{D}}_n^m$: $U(m) \otimes U(n)$
- $\underline{\underline{S}} \in U(m) \otimes U(n) \Rightarrow (\underline{\underline{U}}\,\underline{\underline{S}})\, \underline{\underline{D}}_n^m\, (\underline{\underline{U}}\,\underline{\underline{S}})^{+} = \underline{\underline{U}}\, \underline{\underline{D}}_n^m\, \underline{\underline{U}}^{+}$
- K is isomorphic to the "Grassmannian" coset space

$$G_{m+n,m}(C) = U(m+n) / U(m) \otimes U(n)$$

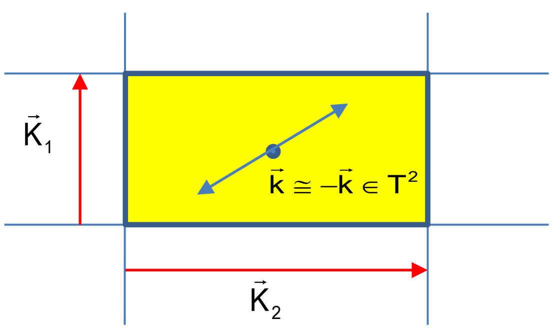
- Topological quantum numbers for isolators :

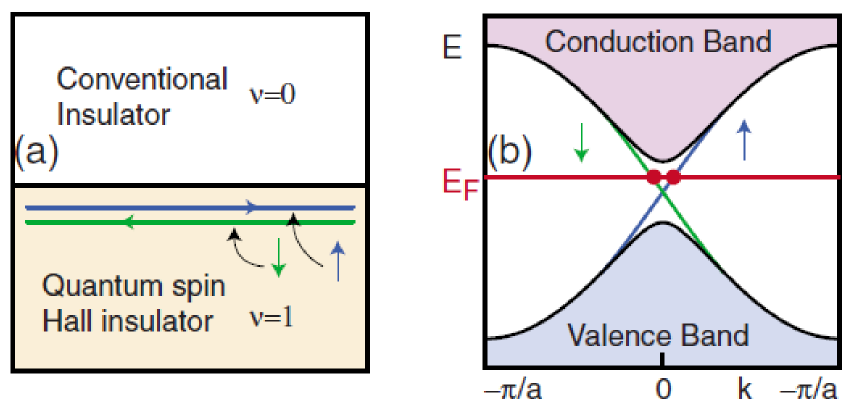
$$\pi(T^d, G_{m+n,m}(C)) = Z \ \text{ for } d = 2,3$$



The spin quantum Hall effect

- Restriction on the base manifold by time reversal symmetry : $\vec{k} \cong -\vec{k}$
- New base manifold $T^d / \cong$
- New classification by
  $\nu \in \pi\left(T^d / \cong, G_{m+n,m}(\mathbb{C})\right) = Z_2 = \{0,1\},\ d = 2,3$
- HgTe/CdTe quantum well structures in 2d
- Spin polarized edge currents
- $Bi_{1-x}Sb_x$, Bi_2S_3, Bi_2Te_3 in 3d, surface currents





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7. Summary

Summary

- Up to 1980: Quantum numbers based on symmetry
- Easy to break, lift of degeneracies
- Since 1980: Topological quantum numbers
- New states of quantum matter
- Robust

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